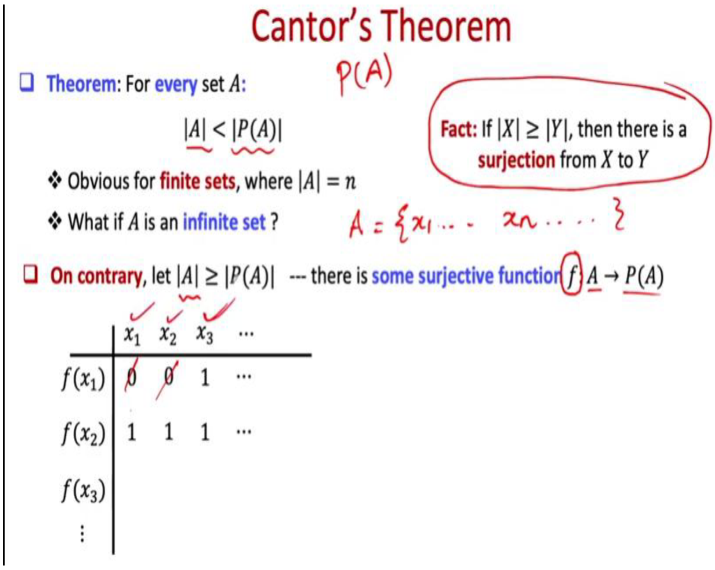

So Cantor proved a very interesting result as well. He showed that you take any set A then the cardinality of that set is strictly less than the cardinality of its power set. So remember the notation P(A) denotes the power set of A. Where the power set is the set of all subsets of that set. So of course this statement is true if your set of A is finite namely if your set A has n number of elements then its power set will have 2^n elements.

And we can always prove that n is always strictly less than 2^n. What if A is an infinite set can we conclude that this theorem is true that is even for infinite set and Cantor showed yes, so the proof is again is contradiction and we will run the diagonalization argument here as well. So we will assume that: let the cardinality of the set A be greater than equal to the cardinality of its power set.

Now before proceeding with fact which we will be using in this proof is the following. If you have the sets X and Y and if the cardinality of X is greater than equal to the cardinality of Y then there always exist a surjection from X to Y. This is a very simple fact which you can prove very easily. So I am not going to the proof of that; we are going to utilize this fact in this proof. So I am assuming here that the cardinality of A is greater than equal to the cardinality of its power set.

That means there will be some surjective function from the set A to the power of set of A. I do not know what exactly is the structure of the surjective function but I denote the surjective function by f. So now what I have done here is let the elements of A be x_1, x_2, x_3 and x_n and so on. It is an infinite set, so it has infinitely many elements, so I am assuming that elements of set A can be listed down as x_1, x_2, x_3 and so on. And I have listed down $f(x_1)$, $f(x_2)$, $f(x_3)$ and so on.

So each of the f value is nothing but a subset of A set that means it will be the element of the power set. So depending upon which elements from the set A are present in $f(x_1)$ accordingly I have put the entry 0's and 1's. So for example here I mean to say that $f(x_1)$ it is a set which does not have x_1, it does not have x_2 but it has x_3 and so on. Similarly $f(x_2)$ is a subset which has x_1 , it has x_2, it has $x_3$and I have listed down $f(x_1)$, $f(x_2)$, $f(x_3)$ and so on. That is the interpretation of 0's and 1's in the table,

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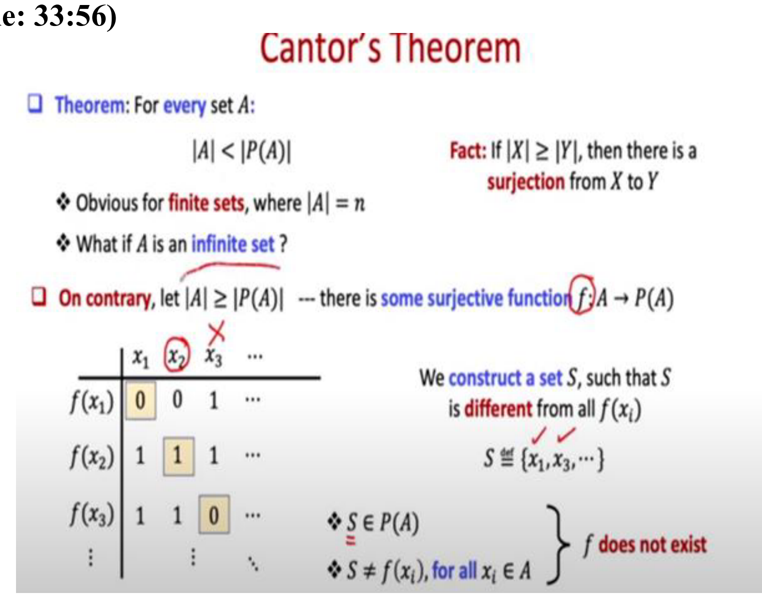

Similarly $f(x_3)$ is the subset which has x_1, which has x_2 but it does not have x_3 and so on. Now what I am going to show is since I am assuming that the function f is surjective function and I have to arrive at a contradiction. I will show that this f actually does not exist and how do I show that f is does not exist? I have to show that f is actually not a valid surjective function. To do that I will show you a subset which belongs to the powerset namely I will show you a subset of A set which do not have any pre image.

That means it will be different from, $f(x_1)$, it will be different from $f(x_2)$, it will be different from any $f(x_i)$ which shows that the set S, the subset S do not have any pre image, hence contradicting that function f is a subjective function. So how do I construct that subset S; again I run the diagonalization argument, so I focus on the diagonal entries here and the elements of my set S will be constructed depending upon the diagonal entries.

So in this example so the first diagonal entry is 0 so I will include x_1 because here the entry is 0 so I will flip it and I will make it 1 that means I have to include x_1. The second entry is 1 along the diagonal 1 so I will flip it and make it 0 that means I have to exclude x_2. The third entry along the diagonal is 0 so I will flip it and will make it 1 that means I have to include x_3 and so on. So that is the way I have constructed set S here.

So now you can check here that indeed the set S I have constructed will be an element of the power set because it is a subset, it will have some of the element from the A set. I am not taking the element in the S set some from outside. So that is why it will be the element of the power set. But you can check here that the S will be not equal to $f(x_1)$ that means the set S will have at least 1 element which is not there in $f(x_1)$.

So for example x_1 is present in S but x_1 was not present in $f(x_1)$. Similarly the set S will be different from $f(x_2)$ why? Because $f(x_2)$ has x_2 but I have not included x_2 in S. Similarly $f(x_3)$ will be different from set S why? Because $f(x_3)$, does not have x_3 but I have included x_3 in S. So the way I have constructed the set S it will be different from the image of $f(x_1)$, $f(x_2)$, $f(x_3)$ and so on.

That means S will not be the image of any x_i and hence S is not a valid surjective function. That means whatever I assumed here namely I have assumed existence of the surjective function from the set A to its power set which is not a valid assumption. This is not a valid assumption because I made here a wrong assumption that the cardinality of the set A is greater than equal to the cardinality of its power set. So that means indeed the statement in this theorem is the correct statement.

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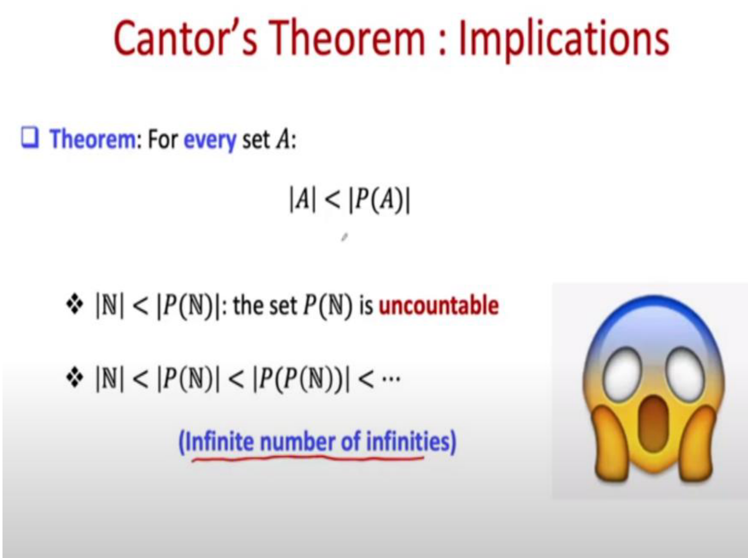

So what is the implication of the Cantor's theorem it has a very beautiful implication. So this is the statement of the cantor's theorem. So if I apply it over the set A being the set of positive integers or the set of natural number then I obtain the fact that the cardinality of the set of natural number is strictly less than the cardinality of its power set. That means the cardinality of the set of natural number is $\aleph_0$.

But what I am showing here is that its $\aleph_0$ is strictly less than the cardinality of the power set of the natural number. That means the power set of the set of natural number is uncountable. Now if I treat the power set of natural number as the set A then the power set of this power set will have more cardinality and this process will keep on going forever. So what basically Cantor showed is that there are infinite number of infinities.

You do not have only one infinity so $\aleph_0$ is one of the infinity it is one of the infinite quantities. But you can have now an infinite number of infinities because now you have a hierarchy of infinite quantities. So that is the very interesting fact about the cardinality of infinite sets. So that brings me to the end of this lecture. Just to summarize this lecture we saw some uncountable sets and we proved that those sets are uncountable by using Cantor's diagonalization argument. Thank you.